

A VOLUME STABILITY THEOREM ON TORIC MANIFOLDS WITH POSITIVE RICCI CURVATURE

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ABSTRACT. In this short note, we will prove a volume stability theorem which says that if an n -dimensional toric manifold M admits a \mathbb{T}^n invariant Kähler metric ω with Ricci curvature no less than 1 and its volume is close to the volume of \mathbb{CP}^n , M is bi-holomorphic to \mathbb{CP}^n .

1. Introduction

To understand the geometry of manifolds under various curvature conditions is a fundamental question. In Riemannian geometry, we have Bishop-Gromov's volume comparison if the Ricci curvature of the manifold is bounded from below. Using this theorem and some techniques in comparison geometry, Colding proved the following result ([5]):

Theorem 1.1. *Given $\epsilon > 0$, there exists $\delta = \delta(n, \epsilon) > 0$ such that, if an n -dimensional manifold M has $\text{Ric}_M \geq n - 1$ and $\text{Vol}(M) > \text{Vol}(\mathbb{S}^n) - \delta$, then $d_{GH}(M, \mathbb{S}^n) < \epsilon$.*

Here d_{GH} denotes the Gromov-Hausdorff distance between Riemannian manifolds. By another theorem of Colding (see the appendix in [4]), we know that M is in fact diffeomorphic to \mathbb{S}^n .

It is a natural question that how to get a more useful version of Bishop-Gromov's volume comparison theorem in Kähler geometry and how to state a theorem analogous to the above one. Because we have more structures on the manifold, the volume comparison with space form is not sharp: see [8] for an improvement of local volume comparison for Kähler manifolds with Ricci curvature bounded from below. More recently, Berman and Berndtsson considered toric manifolds with positive Ricci curvature in [2] and [3], and they proved

Theorem 1.2. *Suppose that (M, ω) is smooth n -dimensional toric variety with \mathbb{T}^n invariant Kähler form ω such that $\text{Ric } \omega \geq \omega$, then we have*

$$(1) \quad \text{Vol}(M) \leq \text{Vol}(\mathbb{CP}^n).$$

In fact, their theorem holds if the manifold admits a \mathbb{C}^* action with finite fixed points and the metric is \mathbb{S}^1 invariant (see [3]). The theorem of Berman and Berndtsson partially answered a conjecture in [10]:

Conjecture 1.3. *Any n -dimensional toric Fano manifold X that admits a Kähler-Einstein metric has anticanonical degree $(-K_X)^n \leq (n+1)^n$, with equality only for \mathbb{CP}^n .*

In this short note, we will determine when the equality holds in Theorem 1.2 and in particular we give a complete answer to Conjecture 1.3. More precisely, we can prove a rigidity and stability theorem as follows:

Theorem 1.4. *The equality in Theorem 1.2 holds if and only if (M, ω) is isometric to $(\mathbb{CP}^n, \omega_{FS})$. Moreover, there exists a positive number ϵ which depends only on n such that if M is a toric manifold with a \mathbb{T}^n invariant metric ω satisfying $\text{Ric } \omega \geq \omega$ and*

$$(2) \quad \text{Vol}(M) \geq (1 - \epsilon) \text{Vol}(\mathbb{CP}^n),$$

M is bi-holomorphic to \mathbb{CP}^n .

In [3], Berman and Berndtsson applied a Moser-Trudinger typed inequality established in [1] to prove Theorem 1.2. But so far we can't prove the rigidity using this analytic method. Inspired by the combinatoric proof by Bo'az Klartag for the Kähler-Einstein case in [3], we will apply the Grunbaum's inequality ([7]) to prove our theorem. In order to use this inequality we should know the position of the barycenter of the moment polytope of (M, ω) . We will use the Ricci curvature condition to achieve this. More detailed analysis gives us the rigidity and stability.

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2. Preliminaries

At first, we give some basic materials of toric manifolds which are used in our proof. Here a toric manifold means a Kähler manifold (M, ω) containing $(\mathbb{C}^*)^n$ as a dense subset such that the standard action of $(\mathbb{C}^*)^n$ on itself extends to a holomorphic action on M . In general we suppose that the metric is \mathbb{T}^n invariant and we can consider the moment map of $(M, \omega, \mathbb{T}^n)$.

Definition 2.1. A polytope $P \subseteq \mathbb{R}^n$ is called a Delzant polytope if each vertex is contained in exactly n facets, and the normals of the n facets containing a given vertex form an integral basis of \mathbb{Z}^n .

The image of the moment map above should be a Delzant polytope according to a theorem of Delzant ([6]):

Theorem 2.2. *Each Delzant polytope gives rise to a symplectic manifold (M, ω) with an action of \mathbb{T}^n that preserves ω , and all such symplectic manifolds arise this way.*

Using the embedding of $(\mathbb{C}^*)^n$ in M , we set:

$$(3) \quad \iota : (\mathbb{C}^*)^n \rightarrow M,$$

$$(4) \quad \iota^* \omega = \sqrt{-1} \partial \bar{\partial} u.$$

. In toric coordinates: $\exp(x_i) = |z_i|^2$ (z_i are holomorphic coordinates in $(\mathbb{C}^*)^n$), the invariance of ω means u is function of x_i . Then the image of $\nabla u = (\frac{\partial u}{\partial x_i})_{1 \leq i \leq n}$ will be a moment map of (M, ω) .

Given a toric manifold (M, ω) , two moment maps may differ by a constant vector. When we choose a basis of group $(\mathbb{C}^*)^n$, these two moment polytopes differ by a translation. A change of basis of group $(\mathbb{C}^*)^n$ corresponds to a change of the integral basis of \mathbb{Z}^n , so it transforms Delzant polytopes to Delzant polytopes. The polytope also changes if we choose another \mathbb{T}^n invariant Kähler metric on M with the same complex structure, i.e. we choose another symplectic form compatible with the fixed complex structure. This can be described in the following way: we denote the moment polytope by

$$(5) \quad P = \{x | \langle l_i, x \rangle \geq \lambda_i, 1 \leq i \leq N, x \in \mathbb{R}^n, l_i \in \mathbb{Z}^n, \lambda_i \in \mathbb{R}\}.$$

Then only λ_i ($1 \leq i \leq n$) change while l_i ($1 \leq i \leq n$) remain the same since they are just related to the complex structure (see [9][11]). Using the description above, changing the symplectic form corresponds to changing the potential function u on \mathbb{R}^n .

When the manifold is a Fano variety with $\omega \in 2\pi c_1(M)$, we can get a moment polytope P such that λ_i are all equal to -1 . It's can be realized in the following way (see [9]): choose a potential u of ω such that

$$(6) \quad |\ln \det u_{ij} + u| \text{ is bounded in } \mathbb{R}^n,$$

then the image of ∇u will be such a polytope P with $\lambda_i = -1$ ($1 \leq i \leq n$).

Because the normal vectors of the facets passing any point form an integral basis, we can do a coordinate transformation to change these vectors to the standard basis $e_k = (0, 0, \dots, 1, 0, \dots, 0)$ with 1 placed at position k . We can write this transformation as follows: choosing a vertex $p \in P$ with l_i ($1 \leq i \leq n$) as normal vectors of the facets passing p , we can form an affine map:

$$(7) \quad x \mapsto (\langle l_i, x \rangle)_{1 \leq i \leq n},$$

which transforms p to $(-1, -1, \dots, -1)$ and the polytope to

$$(8) \quad \tilde{P} = \{x | \langle \tilde{l}_i, x \rangle \geq -1, 1 \leq i \leq N, x \in \mathbb{R}^n, \tilde{l}_i \in \mathbb{Z}^n, \tilde{l}_k = e_k, 1 \leq k \leq n\}.$$

There are only finite many such polytopes in a given dimension.

According to Mabuchi's theorem ([9]), we know that for a Kähler-Einstein manifold, the origin is the barycenter of P . We will prove a similar property

of the barycenter of the moment map of a toric manifold admitting ω with $\text{Ric } \omega \geq \omega$.

3. Proof of the theorem

At first we give a lemma which deals with the volume of some specific kind of polytopes. Let Q be the simplex spanned by

$$(9) \quad (n+1, 0, 0, \dots, 0), (0, n+1, 0, 0, \dots, 0), \dots, (0, 0, \dots, 0, n+1).$$

Recall that the Grunbaum's inequality ([7]) says that if P is a convex body, and K denotes the intersection of P with an affine half-space defined by one side of a hyperplane H passing through the barycenter of P , then

$$(10) \quad \text{Vol}(P) \leq \left(\frac{n+1}{n}\right)^n \text{Vol}(K).$$

Let F denote the simplex spanned by $(n, 0, 0, \dots, 0), (0, n, 0, 0, \dots, 0), \dots, (0, 0, \dots, 0, n)$. We have the following lemma.

Lemma 3.1. *If the barycenter of a polytope P in the first quadrant lies inside F ,*

$$(11) \quad \text{Vol}(P) \leq \text{Vol}(Q).$$

Moreover if the equality holds, P is coincident with Q .

Proof. The first statement can be seen from Grunbaum's theorem above: the corresponding $K \subseteq F$. For the second statement, let $X = P \setminus Q, Y = Q \setminus P$ and choose a coordinate system s_i with the barycenter as the origin and $(1, 1, \dots, 1)$ as the first axis. Then we have

$$(12) \quad \int_P s_1 dV \leq \int_Q s_1 dV = 0, \int_X s_1 dV \leq \int_Y s_1 dv.$$

But since

$$(13) \quad s_1(x) \geq s_1(y) \text{ for } x \in X \text{ and } y \in Y,$$

both X and Y should be empty. \square

In order to apply this lemma to the moment polytope P of (M, ω) , we should know how to place P and where the barycenter is. We are going to use the toric structure on M and explore the Ricci curvature condition.

Under the condition of theorem 1.4, we can write $\text{Ric } \omega = \omega + \beta$ where β is a semi-positive 1-1 form. In $(\mathbb{C}^*)^n$, we can choose u such that $\omega = \sqrt{-1} \partial \bar{\partial} u$. In toric coordinates: $|z_i|^2 = \exp(x_i)$, we set:

$$(14) \quad v = -\ln \det u_{ij} - u.$$

Using the formula of Ricci curvature and $\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} = \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{1}{z_i} \frac{1}{\bar{z}_j}$, we see that

$$(15) \quad \sqrt{-1} \partial \bar{\partial} v = \text{Ric } \omega - \omega = \beta.$$

As β is semi-positive, v is a convex function.

From the following equalities:

$$(16) \quad \ln \det(u+v)_{ij} + u + v = \ln \det(u+v)_{ij} - \ln \det u_{ij} = \ln \frac{(\text{Ric } \omega)^n}{\omega^n},$$

we know that $\ln \det(u+v)_{ij} + u + v$ is bounded, so $\nabla(u+v)$ will be a moment map of $(M, \text{Ric } \omega)$. Denote the image of $\nabla(u+v)$ by L . As illustrated in section 2, we can suppose that $(-1, -1, \dots, -1)$ is a vertex of L and the facets passing it are parallel to coordinate hyperplanes respectively:

$$(17) \quad L = \{y | \langle l_i, y \rangle \geq -1, 1 \leq i \leq N, y \in \mathbb{R}^n, l_i \in \mathbb{Z}^n, l_k = e_k, 1 \leq k \leq n\}.$$

The gradient of u will be a moment of (M, ω) . We denote the image of ∇u by P . Without changing $u+v$, we can add a linear function to u and subtract the same one from v . This corresponds to a translation of P . As we have said above, P can be obtained from L by parallel movement of the facets. So we can translate P so that $(-1, -1, \dots, -1)$ is a vertex of P and the facets passing this vertex are parallel to coordinate hyperplanes like L :

$$(18) \quad P = \{y | \langle l_i, y \rangle \geq \lambda_i, 1 \leq i \leq N, y \in \mathbb{R}^n, l_i \in \mathbb{Z}^n, l_k = e_k, \lambda_k = -1, 1 \leq k \leq n\}.$$

Such a pair of polytopes (P, L) is called an adapted pair of (M, ω) .

Lemma 3.2. *For an adapted pair (P, L) , the coordinates of the barycenter of P are all nonpositive.*

Proof.

$$(19) \quad \lim_{x_i \rightarrow -\infty} \frac{\partial v}{\partial x_i} = \lim_{x_i \rightarrow -\infty} \frac{\partial(u+v)}{\partial x_i} - \lim_{x_i \rightarrow -\infty} \frac{\partial u}{\partial x_i} = (-1) - (-1) = 0$$

for any i and fixed $x_j (1 \leq j \leq n, j \neq i)$. Because v is convex function we know that all the partial derivatives of v are nonnegative. Denote the coordinates of the barycenter by a_i , we have

$$(20) \quad \det u_{ij} = \exp(-u-v), \frac{\partial v}{\partial x_i} \geq 0,$$

$$(21) \quad a_i = \int_P y_i dV = \int_{\mathbb{R}^n} \frac{\partial u}{\partial x_i} \det u_{ij} dx \leq \int_{\mathbb{R}^n} \frac{\partial(u+v)}{\partial x_i} \exp(-u-v) dx = 0.$$

The last inequality is the statement of the lemma. \square

Proof of Theorem 1.4. Using the notations above, we do a translation which moves $(-1, -1, \dots, -1)$ to the origin. Then P will be a polytope inside the first quadrant with barycenter inside F by the second lemma. The rigidity follows from this together with the assumption that $\text{Vol}(P) = \text{Vol}(Q)$ by the first lemma.

Now we consider the stability. Suppose the statement doesn't hold, then there is a sequence of manifolds $(M_i, \omega_i)(i = 1, 2, 3\dots)$ with volume converging to $\text{Vol}(\mathbb{CP}^n)$ and none of them is holomorphic to \mathbb{CP}^n .

Construct adapted pairs (P_i, L_i) of $(M_i, \omega_i)(i = 1, 2, 3\dots)$. Because there are only finitely many such L , one of them appears infinitely times. We denote it by B and select these P_i corresponding to B . These P_i as moment polytopes of different symplectic classes can be obtained from B by parallel movement of B 's facets of towards the interior. So P_i can be determined by N real numbers λ_i such that n of them are always -1 . This gives us a correspondence:

$$(22) \quad P_i \leftrightarrow \lambda^{(i)} \in \mathbb{R}^{N-n}.$$

Because P_i are inside B , these vectors in \mathbb{R}^{N-n} are bounded. We can choose a convergent subsequence, and the limit corresponds to a polytope P_∞ . $\text{Vol}(P_\infty) = \text{Vol}(Q)$ and the coordinates of the barycenter of P_∞ are all nonpositive. According to the first lemma, P_∞ should be isomorphic to Q by a translation. We are going to show that $B = P_\infty$: Since $P_i \subseteq B$, we have $P_\infty \subseteq B$. If $P_\infty \subsetneq B$, the integral points in the interior of the facet of P_∞ opposite $(-1, -1, \dots, -1)$ will be contained in the interior of B . But there is only one integral point in the interior of B , so we must have $B = P_\infty$.

We assumed that M_i are not holomorphic to \mathbb{CP}^n , but now B just differs from Q by a translation. It follows that these M_i are all holomorphic to \mathbb{CP}^n . It's a contradiction, so our theorem is proved. \square

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